9.1 Vector Properties and Operations Notes

Scalar Quantities: Quantities such as length, area, volume, temperature, and time, which have magnitude (size), but no direction.

Vector Quantities: Quantities that involve both a magnitude and direction, such as velocity, acceleration, and force. These quantities can be represented by **directed line segments** called **vectors**.

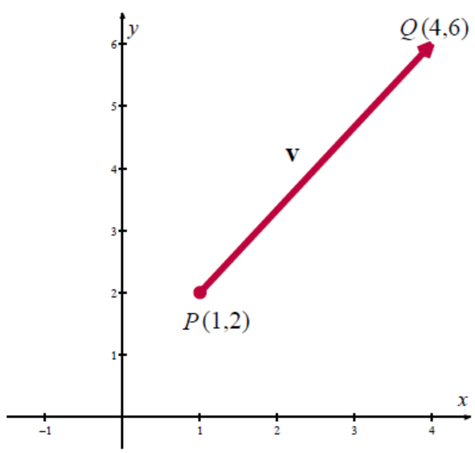
The Geometry of Vectors

A **vector** is represented geometrically as a directed line segment where the **magnitude** of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrowhead at one endpoint of the segment. A vector has an **initial point**, where it begins, and a **terminal point**, indicated by an arrowhead, where it ends. There are various symbols that distinguish vectors from other quantities:

- Lower case type, boldfaced or with an arrow on top, such as $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$, $\vec{v}$, $\vec{u}$, $\vec{w}$.[2]
- Given an initial point P and a terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrow on top is what indicates that it is not just a line, but a directed line segment.

Below is a typical vector $\mathbf{v}$ with endpoints $P(1,2)$ and $Q(4,6)$. The point P is the initial point, or tail, of $\mathbf{v}$ and the point Q is the terminal point, or head, of $\mathbf{v}$. Since we can reconstruct $\mathbf{v}$ completely from P

and Q, we write $\mathbf{v} = \overrightarrow{PQ}$, where the order of points P (initial point) and Q (terminal point) is important. (Think about this before moving on.)

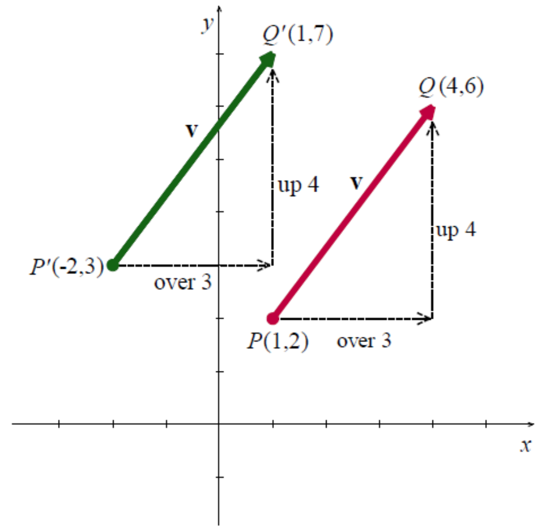


$$\mathbf{v} = \overrightarrow{PQ}$$

While it is true that P and Q completely determine $\mathbf{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\mathbf{v}$ is considered to be the same vector as $\mathbf{v}$, regardless of its initial point. In the case of our vector $\mathbf{v}$ above, any vector which moves three units to the right and four up[3] from its initial point to arrive at its terminal point is considered the same vector as $\mathbf{v}$.

The Component Form of a Vector

The notation we use to capture this idea is $\mathbf{v} = \langle 3,4 \rangle$, the **component form** of the vector, where the first number, 3, is called the x-component of $\mathbf{v}$ and the second number, 4, is called the y-component of $\mathbf{v}$. If we wanted to reconstruct $\mathbf{v} = \langle 3,4 \rangle$ with initial point $P'(-2,3)$ then we would find the terminal point of $\mathbf{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$, as seen to the right.



$\mathbf{v} = \langle 3,4 \rangle$

The component form of a vector is what ties these very geometric objects back to Algebra and ultimately Trigonometry. We generalize our example in the following definition.

Definition. Suppose $\mathbf{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\mathbf{v}$ is given by

$$\mathbf{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

Example 9.1.1. Consider the vector whose initial point is $P(2,3)$ and terminal point is $Q(6,4)$. Write $\mathbf{v} = \overrightarrow{PQ}$ in component form.

Solution. Using the definition of component form, we get

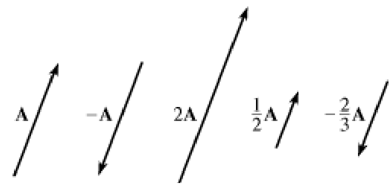
$$\begin{aligned} \mathbf{v} &= \langle 6-2, 4-3 \rangle \\ &= \langle 4,1 \rangle \end{aligned}$$

□

Equal Vectors: Vectors with the same magnitude and direction. They do not have to be in the same place.

Zero Vector: A vector with no magnitude and no direction. It is denoted by $\mathbf{0}$.

Scalar Multiplication: For any scalar k and vector **A,** k**A** is a vector with magnitude $|k|$ times the magnitude of **A**. If $k > 0$, then the direction of $k\mathbf{A}$ is the same as the direction of **A**. If $k < 0$, the direction of $k\mathbf{A}$ is opposite to the direction of **A**. If $k = 0$, then $k\mathbf{A} = \mathbf{0}$.



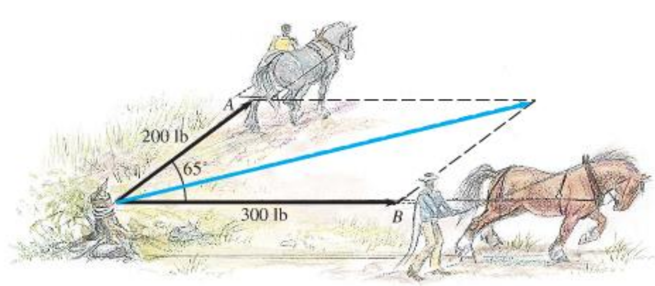
Definition. If k is a real number and $\mathbf{v} = \langle v_1, v_2 \rangle$, we define $k\mathbf{v}$ by

$$k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$$

Theorem 9.2. Properties of Scalar Multiplication.

- **Associative Property:** For every vector $\mathbf{v}$ and scalars k and r, $(kr)\mathbf{v} = k(r\mathbf{v})$.
- **Identity Property:** For all vectors $\mathbf{v}$, $1\mathbf{v} = \mathbf{v}$.
- **Additive Inverse Property:** For all vectors $\mathbf{v}$, $-\mathbf{v} = (-1)\mathbf{v}$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every vector $\mathbf{v}$ and scalars k and r, $(k + r)\mathbf{v} = k\mathbf{v} + r\mathbf{v}$.
- **Distributive Property of Scalar Multiplication over Vector Addition:** For all vectors $\mathbf{v}$ and $\mathbf{w}$ and scalars k, $k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w}$.
- **Zero Product Property:** If $\mathbf{v}$ is a vector and k is a scalar, then $k\mathbf{v} = \mathbf{0}$ if and only if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

Suppose two draft horses are pulling on a tree stump with forces of 200 pounds and 300 pounds, with an angle of 65° between the forces. If **A** and **B** had the same direction, then there would be a total force of 500 pounds acting on the stump, but the total force is less because of the angle between the forces. By the **parallelogram law**, the force acting along the diagonal of the parallelogram, with a magnitude equal to the length of the diagonal, has the same effect on the stump as the two forces **A** and **B**. The force **A + B** acting along the diagonal is called the **sum** or **resultant** of **A** and **B**.



Vector Addition: To find the resultant or sum **A + B** of any vectors **A** and **B**, position **B** (without changing its magnitude or direction) so that the initial point of **B** coincides with the terminal point of **A**. The vector that begins at the initial point of **A** and ends at the terminal point of **B** is the vector **A + B**. For every vector **A**, there is a vector **– A**, with the same magnitude as **A**, but the opposite direction. For any two vectors **A** and **B**, **A – B = A + (– B)**.

Definition. Suppose $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$. The vector $\mathbf{v}+\mathbf{w}$ is defined by

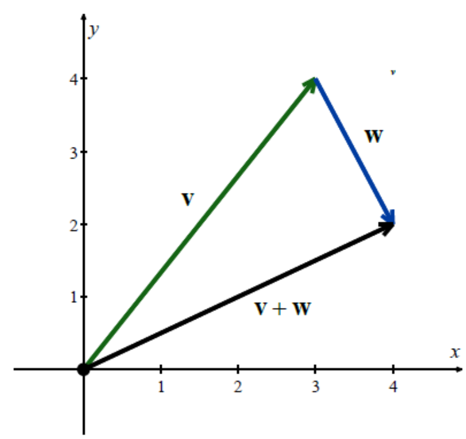
$$\mathbf{v}+\mathbf{w}=\langle v_1+w_1,v_2+w_2\rangle$$

Example 9.1.3. Let $\mathbf{v}=\langle 3,4\rangle$ and suppose $\mathbf{w}=\overrightarrow{PQ}$ for $P(-3,7)$ and $Q(-2,5)$. Find $\mathbf{v}+\mathbf{w}$ and interpret this sum geometrically.

Solution. Before adding the vectors using the definition, we need to write $\mathbf{w}$ in component form. We get $\mathbf{w}=\langle -2-(-3),5-7\rangle=\langle 1,-2\rangle$, and then

$$\begin{aligned}\mathbf{v}+\mathbf{w}&=\langle 3,4\rangle+\langle 1,-2\rangle\\&=\langle 3+1,4+(-2)\rangle\\&=\langle 4,2\rangle\end{aligned}$$

To visualize this sum, we draw $\mathbf{v}$ with its initial point at $(0,0)$, for convenience, so that its terminal point is $(3,4)$. Next, we graph $\mathbf{w}$ with its initial point at $(3,4)$. Moving one to the right and two down, we find the terminal point of $\mathbf{w}$ to be $(4,2)$. We see that the vector $\mathbf{v}+\mathbf{w}$ has initial point $(0,0)$ and terminal point $(4,2)$ so its component form is $\langle 4,2\rangle$ as required.

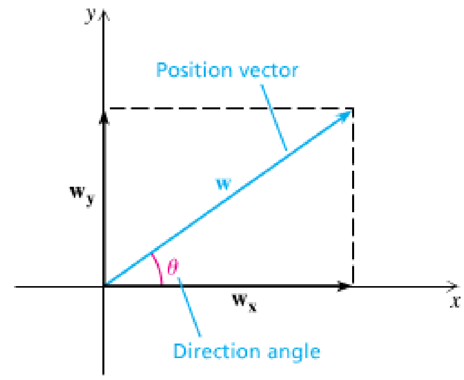


□

Theorem 9.1. Properties of Vector Addition.

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.
- **Associative Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
- **Identity Property:** The vector $\mathbf{0}$ acts as the additive identity for vector addition. That is, for all vectors $\mathbf{v}$, $\mathbf{v}+\mathbf{0}=\mathbf{0}+\mathbf{v}=\mathbf{v}$.
- **Inverse Property:** Every vector $\mathbf{v}$ has a unique additive inverse, denoted $\textbf{-v}$. That is, for every vector $\mathbf{v}$, there is a vector $\textbf{-v}$ so that $\mathbf{v}+(\textbf{-v})=(\textbf{-v})+\mathbf{v}=\mathbf{0}$.

Any nonzero vector **w** is the sum of a **horizontal component**, $\mathbf{w_x}$, and a **vertical component**, $\mathbf{w_y}$. If a vector **w** is placed in a rectangular coordinate system so that its initial point is the origin, then **w** is called a **position vector**. The angle θ formed by the positive *x*-axis and a position vector is the **direction angle** for the position vector.



If the vector **w** has magnitude *r*, direction angle θ, horizontal component $\mathbf{w_x}$, and vertical component $\mathbf{w_y}$, then we get

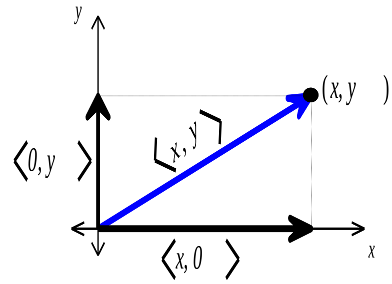
$$\cos\theta = \frac{|\mathbf{w_x}|}{r} \text{ and } \sin\theta = \frac{|\mathbf{w_y}|}{r} \text{ or } |\mathbf{w_x}| = |r\cos\theta| \text{ and } |\mathbf{w_y}| = |r\sin\theta|.$$

Examples: Find the magnitude of the horizontal and vertical components for each vector **v** with the given magnitude and direction angle θ. Round to the nearest tenth.

a) $|\mathbf{v}| = 5.6,\ \theta = 22°$

b) $|\mathbf{v}| = 445,\ \theta = 211.1°$

Component Form: The notation $\langle x, y\rangle$ is used to define a position vector with terminal point (x, y). This is called component form because the horizontal component is $\langle x, 0\rangle$ and its vertical component is $\langle 0, y\rangle$.



The magnitude of the vector $\mathbf{v} = \langle x, y\rangle$ is $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$. To find the direction angle, use $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

If a vector has magnitude r and direction angle θ, its component form is $\langle r\cos\theta,\ r\sin\theta\rangle$.

Examples: Find the magnitude and direction angle of each vector.

a) $\mathbf{v} = \langle 2, -6\rangle$

b) $\mathbf{v} = \langle -3, 2\rangle$

c) $\mathbf{v} = \langle -4, -5\rangle$

Examples: Find the component form for each vector **v** with the given magnitude and direction angle θ.

a) $|\mathbf{v}| = 12,\ \theta = 45°$

b) $|\mathbf{v}| = 50,\ \theta = 120°$

If $\mathbf{A} = \langle a_1, a_2 \rangle$, $\mathbf{B} = \langle b_1, b_2 \rangle$, and k is a scalar, then

Vector Arithmetic:

1. $k\mathbf{A} = \langle ka_1, ka_2 \rangle$ **Scalar Product**
2. $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$ **Vector Sum**
3. $\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$ **Vector Difference**
4. $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2$ **Dot Product**

Examples: Let $\mathbf{w} = \langle -1, -3 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$. Perform the operations indicated.

a) $\mathbf{w} - \mathbf{v}$ b) $-8\mathbf{v}$ c) $3\mathbf{w} + 4\mathbf{v}$ d) $\mathbf{w} \cdot \mathbf{v}$